Chapter 3 Elementary properties and examples of analytic functions. 1.

§ 1 Power Series.

Def. . If a_n is in $\mathbb{C}$ for every $n \geq 0$, then the series converges to $z \in \mathbb{C}$

iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$

s.t. $\left| \sum_{n=0}^{m} a_n - z \right| < \varepsilon$ whenever $m \geq N$

Accordingly, we denote $z = \sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n$.

- The series converges absolutely if $\Sigma |a_n|$ converges.

Prop 1.1. If Σa_n converges absolutely, then Σa_n converges.

Pf. Let $\varepsilon > 0$ and set $z_n = a_1 + a_2 \cdots + a_n$.

Since $\Sigma |a_n|$ converges,

$\exists N \in \mathbb{N}$ s.t. $\left| \sum_{n=0}^{N-1} |a_n| - \sum_{n=0}^{\infty} |a_n| \right| < \varepsilon$

$= \sum_{n=N}^{\infty} |a_n|$.

Thus, whenever $m > k \geq N$,

$$|z_m - z_k| = \left| \sum_{n=k+1}^{m} a_n \right| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=k+1}^{\infty} |a_n| < \varepsilon.$$

That is, $\{z_n\}$ is a Cauchy sequence and so

$\exists z \in \mathbb{C}$ s.t. $z = \lim z_n$.

Hence $z = \sum_{n=0}^{\infty} a_n$. □

Def Let $\{a_n\}$ be a sequence in $\mathbb{R}$.

Define $\liminf a_n = \lim_{n \to \infty} \left(\inf \{a_n, a_{n+1}, \cdots\} \right)$

$\limsup a_n = \lim_{n \to \infty} \left(\sup \{a_n, a_{n+1}, \cdots\} \right)$

Sometime, we also write

$\liminf a_n = \underline{\lim}\, a_n$

$\limsup a_n = \overline{\lim}\, a_n$.

<u>Prop</u> If $\{a_n\}$ is a convergent sequence in $\mathbb{R}$ and $a = \lim a_n$,
then $a = \liminf a_n = \limsup a_n$.

<u>Prop</u> $\liminf a_n \le \limsup a_n$ for any sequence in $\mathbb{R}$.

- A power series about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$.

<u>Ex.</u> If $|z| < 1$, then

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

$$\sum_{n=0}^{m} z^n = 1 + z + z^2 \cdots + z^m = \frac{1-z^{m+1}}{1-z} \longrightarrow \frac{1}{1-z}$$

$|z|<1 \Rightarrow |z|^{m+1} \to 0$

<u>Thm 1.3</u> For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$,
define the number R, $0 \le R \le +\infty$, by

$$\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$$

then (a) if $|z-a| < R$,
then the series converges absolutely.

(b) if $|z-a| > R$,
then the terms of the series become unbounded
and so the series diverges.

c) if $0 < r < R$,
then the series converges uniformly on
$\{z : |z-a| \le r\}$.

Moreover, the number R is the only number
having property (a) and (b).

pf: We may suppose $a=0$.

(a) If $|z|<R$, there is an r with $|z|<r<R$.

Thus, $\exists N \in \mathbb{N}$ s.t. $|a_n|^{\frac{1}{n}} < \frac{1}{r}$ for all $n \geq N$. $\frac{1}{r} > \frac{1}{R}$

Then $|a_n| < (\frac{1}{r})^n$ and $|a_n z^n| < (\frac{|z|}{r})^n$.

Thus $\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\sum_{n=N}^{\infty} (\frac{|z|}{r})^n$

$\frac{|z|}{r} < 1$, $\sum_{n=N}^{\infty} (\frac{|z|}{r})^n$ converges.

thus. $\sum_{n=N}^{\infty} a_n z^n$ converges absolutely.

(b) excersise.

(c) Fix $r < R$ and choose $r < \rho < R$

$\exists N$, s.t. $|a_n| < \frac{1}{\rho^n}$ for $n \geq N$. as above.

If $|z| \leq r$, $|a_n z^n| < \frac{r^n}{\rho^n}$. for $n \geq N$.

$\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\sum_{n=N}^{\infty} (\frac{r}{\rho})^n$

Hence $\sum a_n z^n$ converges uniformly on $\{z : |z| \leq r\}$.

RMK. R is called the radius of convergence of the power series.

Prop. 1.4 If $\sum a_n (z-a)^n$ is a given power series with radius of convergence R, then

$$R = \lim \left|\frac{a_n}{a_{n+1}}\right| \text{ if the limit exists.}$$

proof. We may assume $a=0$.

let $\alpha = \lim \left|\frac{a_n}{a_{n+1}}\right|$.

Suppose $|z| < \alpha$ WTS $\sum |a_n z^n|$ converges

Take r s.t. $|z| < r < \alpha$

$\exists N > 0$, s.t. $\left|\frac{a_n}{a_{n+1}}\right| > r$ for $n \geq N$.

$|a_{n+1}| < |a_n| \cdot r$
$n \geq N$

Let $B = |a_N| \cdot r^N$

Then $|a_{N+1} r^{N+1}| \leq \left|\frac{a_N}{r}\right| \cdot r^{N+1} = |a_N| \cdot r^N = B$

$|a_{N+2} r^{N+2}| \leq \left|\frac{a_{N+1}}{r}\right| \cdot r^{N+2} = |a_{N+1} r^{N+1}| = B$

Inductively, $|a_n r^n| \leq B$ for $n \geq N$.

Since $|z| < r$, $|a_n z^n| = |a_n r^n| \cdot \left(\frac{|z|}{r}\right)^n \leq B \left(\frac{|z|}{r}\right)^n$ for $n \geq N$.

So $\sum |a_n z^n|$ converges.

Suppose $|z| > \alpha$.

WTS. $\sum a_n z^n$ diverges.

Take r s.t. $\alpha < r < |z|$.

Since $\alpha = \lim \left|\frac{a_n}{a_{n+1}}\right| < r$,

$\exists N > 0$ s.t. $\left|\frac{a_n}{a_{n+1}}\right| < r$ for $n \geq N$.

Set $B = |a_N r^N| > 0$.

Then $|a_{N+1} r^{N+1}| > \left|\frac{a_N}{r}\right| \cdot r^{N+1} = B$.

$|a_{N+2} r^{N+2}| \geq \left|\frac{a_{N+1}}{r}\right| \cdot r^{N+2} = |a_{N+1} r^{N+1}| \geq B$.

Inductively, $|a_n r^n| \geq B$. for any $n \geq N$.

$|a_n z^n| > |a_n r^n| \geq B \not\to 0$ as $n \to \infty$.

So $\sum a_n z^n$ diverges.

Therefore $R = \alpha$. $\square$

<u>Ex.</u> Consider $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Note $a_n = \frac{1}{n!}$, $\frac{a_n}{a_{n+1}} = n+1 \to +\infty$

So $R = +\infty$

$e^z \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Prop 1.5 Let Σa_n and Σb_n be two absolutely convergent series and put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Then Σc_n is absolutely convergent with the sum

$$\Sigma c_n = \Sigma a_n \cdot \Sigma b_n$$

Prop 1.6. Let $\Sigma a_n (z-a)^n$ and $\Sigma b_n (z-a)^n$ be power series with radius of convergence $\geq r > 0$.

Put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Then both power series $\Sigma (a_n + b_n)(z-a)^n$ and $\Sigma c_n (z-a)^n$ have radius of convergence $r \geq 0$, and

$$\Sigma (a_n + b_n)(z-a)^n = \Sigma a_n (z-a)^n + \Sigma b_n (z-a)^n$$

$$\Sigma c_n (z-a)^n = \Sigma a_n (z-a)^n \cdot \Sigma b_n (z-a)^n$$

for $|z-a| < r$.

§2. Analytic functions.

Def 2.1 If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$, then f is (complex) differentiable at a point $a \in G$ if

$$f'(a) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

- $f'(a)$ will be called the (complex) derivative of f at a.
- If f is differentiable at each point of G, we say f is differentiable on G.
- $f' : G \to \mathbb{C}$

  If f' is continuous, then we say f is continuously differentiable.
- If f' is (complex) differentiable, we say f is twice differentiable.

Inductively, we can define f is infinitely differentiable.

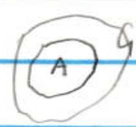
Prop 2.2 If $f: G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.

Def 2.3 A function $f: G \to \mathbb{C}$ is analytic if f is continuously (complex) differentiable.

Chain Rule 2.4 Let f, g be analytic on G and Ω respectively. Suppose $f(G) \subseteq \Omega$. Then $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \quad \text{for any } z \in G.$$

RMK We have defined analytic function f on open set G. For an arbitrary set A, f is analytic on A if f is analytic on some open set G and $A \subseteq G$.



Ex. $f: \mathbb{C} \to \mathbb{C}$ $\quad f(z) = \bar{z}$.

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

$$\frac{\bar{h}}{h} = \begin{cases} 1 & \text{if } h \in \mathbb{R} \\ -1 & \text{if } h \in \sqrt{-1}\,\mathbb{R} \end{cases}$$

Thus f is not (complex) differentiable.

$f(x, y) = x - \sqrt{-1}\, y$ is differentiable w.r.t real variables x and y.

Prop 2.5. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$.

Then

a) For each $k \geq 1$, the series

$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)\, a_n (z-a)^{n-k} \qquad (*)$$

has radius of convergence R.

b) The function f is indefinitely differentiable on $B(a,R)$ and furthermore, $f^{(k)}(z)$ is given by $(*)$ for any $k \geq 1$ and $|z-a| < R$.

c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Again assume that $a = 0$.

(a) It suffices to prove for $k = 1$

Recall $R^{-1} = \limsup |a_n|^{\frac{1}{n}}$

WTS: $R^{-1} = \limsup |n a_n|^{\frac{1}{n}}$

Sufficient to check $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

$$n^{\frac{1}{n}} = e^{\frac{1}{n}\ln n}$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \qquad n^{\frac{1}{n}} \to 1 \text{ as } n \to \infty$$

b) We first prove for $k = 1$.

For $|z| < R$, put $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$S_n(z) = \sum_{j=0}^{n} a_j z^j$, $\quad R_n(z) = \sum_{j=n+1}^{\infty} a_j z^j$

Fix $w \in B(0,R)$ and take r s.t. $|w| < r < R$.

WTS: $f'(w) = g(w)$.

Take $\delta > 0$ s.t. $\overline{B(w,\delta)} \subseteq B(0,r)$.

Let $z \in B(w,\delta)$

$$\frac{f(z)-f(w)}{z-w} - g(w)$$

$$= \left(\frac{S_n(z)-S_n(w)}{z-w} - S_n'(w)\right) + \left(S_n'(w) - g(w)\right) + \left(\frac{R_n(z)-R_n(w)}{z-w}\right)$$

$$\frac{R_n(z)-R_n(w)}{z-w} = \frac{\sum_{j=n+1}^{\infty} a_j\,(z^j - w^j)}{z-w}$$

$$= \sum_{j=n+1}^{\infty} a_j\,(z^{j-1} + z^{j-2}w + \cdots w^{j-1})$$

$$\left|\frac{R_n(z)-R_n(w)}{z-w}\right| \le \sum_{j=n+1}^{\infty} |a_j|\;|z^{j-1}+z^{j-2}w+\cdots w^{j-1}|$$

$$\le \sum_{j=n+1}^{\infty} |a_j|\,(r^{j-1}+r^{j-1}\cdots r^{j-1})$$

$$= \sum_{j=n+1}^{\infty} j\,|a_j|\cdot r^{j-1}$$

Since $r < R$, $\sum_{j=1}^{\infty} |a_j|\,j\,r^{j-1} < +\infty$.

Thus $\exists\, N_1 \in \mathbb{N}$ s.t. $\sum_{j=n+1}^{\infty} j|a_j|\cdot r^{j-1} < \frac{\varepsilon}{3}$ for $n \geq N_1$

So $\left|\frac{R_n(z)-R_n(w)}{z-w}\right| < \frac{\varepsilon}{3}$ for $n \geq N_1$

$$S_n'(w) = \sum_{j=1}^{n} j\,a_j\,w^{j-1} \longrightarrow \sum_{j=1}^{\infty} j a_j\,w^{j-1} = g(w) \quad \text{as } n \to \infty.$$

$\exists\, N_2 \in \mathbb{N}$ s.t. $|S_n'(w) - g(w)| < \frac{\varepsilon}{3}$ for $n \geq N_2$.

Take $n = \max\{N_1, N_2\}$.

Then $|S_n'(w) - g(w)| < \varepsilon$

$\left|\frac{R_n(z) - R_n(w)}{z-w}\right| < \varepsilon$

Since $\lim_{z\to w}\left(\frac{S_n(z) - S_n(w)}{z-w} - S_n'(w)\right) = 0$ for $n = \max\{N_1, N_2\}$,

$\exists\, \delta > 0$, s.t. $\left|\frac{S_n(z) - S_n(w)}{z-w} - S_n'(w)\right| < \varepsilon$ for $|z-w| < \delta$.

So $\left|\frac{f(z) - f(w)}{z-w} - g(w)\right| < \varepsilon$ for $|z-w| < \delta$.

That is

$$f'(w) = \lim_{z\to w}\frac{f(z)-f(w)}{z-w} = g(w).$$

c) $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n z^{n-k}$ by (a)

$f^{(k)}(0) = k!\, a_k$

<u>Cor. 2.9</u> If the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence $R>0$, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic in $B(a, R)$.

<u>Ex.</u> $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is analytic in $\mathbb{C}$.

$(e^z)' = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z$

$e^{a+b} = e^a \cdot e^b$

$e^z \cdot e^{-z} = e^0 = 1$

$\overline{e^z} = e^{\bar z}$

$|e^z| = (e^z \cdot \overline{e^z})^{\frac{1}{2}} = e^{(z+\bar z)/2} = e^{\operatorname{Re} z}$

Def. For $z \in \mathbb{C}$, define

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots + (-1)^n \cdot \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!}$$

RMK. · Convergence radius $R = +\infty$

· $\cos z$, $\sin z$ are extention of $\cos x$, $\sin x$ for $x \in \mathbb{R}$.

Prop

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

$$e^{iz} = \cos z + i \sin z$$

$$\cos^2 z + \sin^2 z = 1$$

$$z = |z| e^{i\theta} \qquad \theta = \arg z.$$

Def. A function f is periodic with period c if $f(z+c) = f(z)$ for all $z \in \mathbb{C}$

E.x. Find the period of e^z.

$$e^{z+c} = e^z \qquad e^c = 1$$

$$c = a + ib. \qquad a, b \in \mathbb{R}.$$

$$e^c = e^a(\cos b + i \sin b) = 1$$

$$a = 0 \qquad b = 2k\pi \qquad \text{for } k \in \mathbb{Z}.$$

$$c = 2\pi k i \qquad k \in \mathbb{Z}$$

$z \to e^z = w$

$z + 2k\pi i \leftarrow w$

$\parallel$

$\log|w| + i \arg w + i 2k\pi$

$e^z : \mathbb{C} \to \mathbb{C}$ is not one-to-one.

It has no inverse functions.

We take a branch of its 'multi-valued inverse'.

Def. If G is an open and connected subset of $\mathbb{C}$,
and $f: G \to \mathbb{C}$ is a continuous function
s.t. $z = \exp f(z)$ for $z \in G$,
then f is a branch of the logarithm.

RMK. $|e^z| = e^{\operatorname{Re} z} > 0$.
$e^z: \mathbb{C} \to \mathbb{C} - \{0\}$.
So we must have $0 \notin G$.

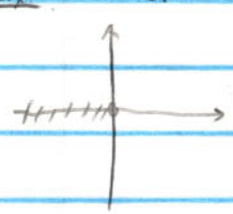
Prop. If $G \subseteq \mathbb{C}$ is open and connected and $f: G \to \mathbb{C}$ is a branch of $\log z$,
then $g: G \to \mathbb{C}$ is a branch of $\log z$.
iff $g(z) = f(z) + 2k\pi i$ for some $k \in \mathbb{Z}$

proof. '$\Leftarrow$' easy.
'$\Rightarrow$' set $h(z) = \frac{1}{2\pi i}(f(z) - g(z))$

$$e^{2\pi i h(z)} = e^{f(z) - g(z)}$$
$$= z/z$$
$$= 1 \qquad \text{for } z \in G$$

Thus $h(z) \in \mathbb{Z}$
Since h is continuous, and G is connected,
h is a constant integer.
So $f(z) - g(z) = 2\pi i h$ for some $k \in \mathbb{Z}$. $\square$

Ex. $G = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$.



$f: G \to \mathbb{C}$
$\forall z \in G, \quad z = re^{i\theta} \quad \theta \in (-\pi, \pi)$
$f(z) = f(re^{i\theta}) \overset{\text{def.}}{=} \log r + i\theta$. principle branch.

Prop. Let G and Ω be open subsets of $\mathbb{C}$.
Suppose that $f: G \to \mathbb{C}$ and $g: \Omega \to \mathbb{C}$ are continuous functions s.t. $f(G) \subseteq \Omega$ and $g(f(z)) = z$ for $z \in G$.
If g is differentiable and $g'(z) \neq 0$,
then f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$
Furthermore, if g is analytic, then f is analytic.

Pf: Fix $a \in G$, and let $h \in \mathbb{C}$ s.t. $h \neq 0$ and $a+h \in G$.
Note that $a = g(f(a))$ and $a+h = g(f(a+h))$
implies that $f(a) \neq f(a+h)$.

$$1 = \frac{g(f(a+h)) - g(f(a))}{h}$$

$$= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \quad \frac{f(a+h) - f(a)}{h}$$

f is continuous $\Rightarrow$ $f(a+h) - f(a) \to 0$ as $h \to 0$

Thus.

$$\lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} = g'(f(a))$$

Thus. $\lim \dfrac{f(a+h) - f(a)}{h}$ exists

and $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{1}{g'(f(a))}$.

$f'(a) = \dfrac{1}{g'(f(a))}$ is continuous.

So f is analytic $\square$

RMK. Any branch of log is analytic.

Cauchy - Riemann equation

$$f : G \to \mathbb{C}$$
$$z \to f(z)$$

$z = x + iy \qquad f(x+iy) = u(x,y) + i\,v(x,y).$

Suppose f is analytic

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

We evaluate the limit in two ways.

① Take $h \in \mathbb{R}$.

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h, y) - u(x,y)}{h} + i\,\frac{v(x+h, y) - v(x)}{h}$$

$$\to u_x(x,y) + i\,v_x(x,y)$$

② Take $h = it$ and $t \in \mathbb{R}$

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x, y+t) - u(x,y)}{it} + i\,\frac{v(x, y+t) - v(x,y)}{it}$$

$$\to -i\,u_y(x,y) + v_y(x,y)$$

Comparing the limit,

$$\begin{cases} u_x(x,y) = v_y(x,y) \\ u_y(x,y) = -v_x(x,y) \end{cases}$$

This is the so-called Cauchy - Riemann equations.

Prop 2.10 G a region (open and connected) in $\mathbb{C}$

$f : G \to \mathbb{C}$ is analytic and $f' \equiv 0$ on G.

Then f is a constant

$f: G \to \mathbb{C}$ $\qquad f = u + iv$.

f is analytic $\Rightarrow$ Cauchy - Riemann equations.

<u>Q</u> How about the converse?

<u>Thm 2.29</u> Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous partial derivatives.
then $f: G \to \mathbb{C}$ defined by $f(z) = u + iv$ is analytic iff u and v satisfy the Cauchy - Riemann equations.

Pf: '$\Rightarrow$' ✓.

'$\Leftarrow$' $\quad f = u + iv$

$$\frac{f(z+s+it) - f(z)}{s+it}$$

$$= \frac{u(x+s, y+t) - u(x,y)}{s+it} + i\,\frac{v(x+s, y+t) - v(x,y)}{s+it}$$

$$u(x+s, y+t) - u(x,y)$$

$$= u(x+s, y+t) - u(x, y+t) + u(x, y+t) - u(x,y)$$

$s_1 \in [0,s]$
$t_1 \in [0,t]$

$$\underset{\text{MVT}}{\overset{\uparrow}{=}} u_x(x+s_1, y+t)\cdot s + u_y(x, y+t_1)\cdot t$$

$$= u_x(x,y)\, s + u_y(x,y)\cdot t + o(s) + o(t)$$

Similarly,

$$v(x+s, y+t) - v(x,y)$$

$$= v_x(x,y)\cdot s + v_y(x,y)\cdot t + o(s) + o(t)$$

$$\frac{f(z+s+it)-f(z)}{s+it} = \frac{u_x(x,y)\cdot s + u_y(x,y)\cdot t + iv_x(x,y)s + iv_y(x,y)t}{s+it} + o(1)$$

$$= \frac{(u_x(x,y)+iv_x(x,y))s + (u_y(x,y)+iv_y(x,y))t}{s+it} + o(1)$$

CR equations.

$$= \frac{(u_x(x,y)+iv_x(x,y))s + (-v_x(x,y)+iu_x(x,y))t}{s+it} + o(1)$$

$$= u_x(x,y) + iv_x(x,y) + o(1)$$

So $\lim_{s+it\to 0} \frac{f(z+s+it)-f(z)}{s+it} = u_x(x,y) + iv_x(x,y)$. $\square$